An Examination of an Iterative Convergence Algorithm for Lagrange Multiplier Dynamic Constraints

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Abstract

This paper describes a proposal for a class project in The Nature of Mathematical Modeling (MAS.864) taught by Professor Neil Gershenfeld, Fall semester 1997.

1 Introduction

Dynamic constraints are a natural way of describing many interesting physical systems. There are two general categories of dynamic constraints discussed in the computer graphics literature: reduced coordinate formulations and Lagrange multiplier dynamic constraints.

Reduced coordinate formulations involve re-parameterizing an unconstrained system with n degrees of freedom as a constrained system with $i < n$ degrees of freedom. This can be quite difficult for arbitrary system of constraints, since there is often no obvious way of re-parameterizing systems involving constraint loops [5, 3].

A more flexible technique is to use Lagrange multiplier dynamic constraints. Lagrange multiplier approaches maintain constraints by introducing constraint forces into the original n degrees of freedom system. Reparameterization is not involved, and the constraints are described in a modular way which makes the construction of arbitrary systems of constraints straightforward. However, Lagrange multiplier techniques involve solving for a vector of Lagrange multipliers λ each time the constraint forces are evaluated, which requires the solution of a system of linear equations. For a system of n constraints, this generally requires solving an equation of the form

$$A\lambda = \mathbf{b},$$

where A is a square symmetric positive definite matrix of dimension n, λ is a vector of n Lagrange multipliers, and $\mathbf{b}$ is a vector of dimension n. Very often A is sparse, in which case conjugate gradient (or other iterative convergence) methods exploiting sparseness may be used. In theory such techniques reduce the order to approximately $O(n^2)$ [8, 3], but since the sparseness of A is not guaranteed solving for λ is generally a $O(n^3)$ operation [3].

Recently, Baraff [3] has described an $O(n)$ Lagrange multiplier solution for acyclic constraint problems. Central to this technique is an always sparse formulation, and a relatively simple $O(n)$ technique for solving the resulting linear system.

Over the last year and a half, I have developed an iterative algorithm for solving arbitrary, cyclic, Lagrange multiplier constraint problems. This algorithm (which is central to my as-yet unpublished M.S. Visualization Science thesis) appears to be $O(n)$ with respect to the number of constraints. However, I do not yet fully understand the convergence properties of this algorithm, which (I believe) is essentially a relaxation process. Gaining a better understanding of this process would be a great benefit to my research.

2 Goals

I have two goals for this project:

1. To apply the principles of analysis I have learned in class to better understand the convergence behavior of the algorithm.

2. To use what I learn in the analysis process to develop an improved algorithm with improved convergence properties.

It is my hope that by studying iterative numerical techniques in general (and relaxation techniques more specifically) I can accomplish my first goal, and that the second will naturally follow from the first. As a starting point, I plan on studying the following references, beginning with the book by William F. Ames: [1, 2, 4, 6, 7].

3 Bibliography

References

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